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Several properties of differential equation with (p, q)-Genocchi polynomials

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Abstract. We construct several differential equations of which are related to (p, q)-Genocchi polynomials in this paper. From these differential equation, we also investigate some relations which are related to Genocchi, q-Genocchi, and (p, q)-Genocchi polynomials.

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1. Introduction

For any $n \in \mathbb{C}$, the (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Wachs and White [9] introduced the (p, q)-numbers in mathematics literature in certain combinatorial problems without any connection to the quantum group related to mathematics and physics literature, see [4], [5], [9].

Definition 1.1 [1], [8]. Let z be any complex numbers with $|z| < 1$. The two forms of (p, q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$

$$E_{p,q}(z)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}.$$

In [2], Corcino made the theorem of (p,q)-extension of binomials coefficients and found various properties which are related to horizontal function, triangular function, and vertical function.

Definition 1.2 [2]. Let $n\geq k$. (p,q)-Gauss Binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!},$$

where $[n]_{p,q}!=[n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$.

Definition 1.3 [1], [8]. (p,q)-derivative operator of any function f, also referred to as the Jackson derivative, is defined the as follows:

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x},\qquad x\neq 0,$$

and $D_{p,q}f(0)=f'(0)$.

Let $p=1$ in Definition 1.3. Then, we can remark

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\qquad x\neq 0,$$

we call D_q is the q-derivative.

Theorem 1.4 [1], [6]. *The operator, $D_{p,q}$, has the following basic properties:*

(i) Derivative of a product

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x) \\ &= g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x). \end{aligned}$$

(ii) Derivative of a ratio

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right)=\frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)}$$
$$=\frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In 2016, Araci et al. [1] introduced a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p,q)-numbers and found some properties and identities. After that, several studies have investigated the special functions for various applications, see [3], [6], [7].

Definition 1.5 [3], [10]. (p,q)-Euler numbers $\mathcal{E}_{n,p,q}$ and polynomials $\mathcal{E}_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\frac{2}{e_{p,q}(t)+1},$$
$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}=\frac{2}{e_{p,q}(t)+1}e_{p,q}(tx).$$

Consider $p=1$ in Definition 1.5. Then, we note

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}\frac{t^n}{[n]_{q}!}=\frac{2}{e_{q}(t)+1},$$
$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_{q}!}=\frac{2}{e_{q}(t)+1}e_{q}(tx),$$

where $\mathcal{E}_{n,q}$ is the q-Euler number and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials.

In Definition 1.5, we can note the Euler numbers and polynomials with condition $p=1, q\to 1$.

Definition 1.6 [3]. (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty}G_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\frac{2t}{e_{p,q}(t)+1},$$
$$\sum_{n=0}^{\infty}G_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}=\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx).$$

Consider $p=1$ in Definition 1.6, we note

$$\sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1},$$
$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx),$$

where $G_{n,q}$ is the q-Genocchi numbers and $G_{n,q}(x)$ is the q-Genocchi polynomials. From Definition 1.6, we can note the Genocchi numbers and polynomials with condition $p=1, q \to 1$.

2. Main results

In this section, we introduce several differential equations which is related to (p,q)-Genocchi polynomials. We also find some relations of Genocchi, q-Genocchi, and (p,q)-Genocchi polynomials using (p,q)-derivative.

Theorem 2.1. *Let* $[n]_{p,q} \neq 0$. *Then, we obtain*

$$D_{p,q,x}^{(k)} G_{n,p,q}(x) = \frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!} G_{n-k,p,q}(p^k x).$$

Proof. From the generating function of (p,q)-Genocchi polynomials, we find

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} G_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} G_{k,p,q} x^{n-k}\right)\frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{1}$$

From (1), we obtain a relation between (p,q)-Genocchi numbers and polynomials as follows.

$$G_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} G_{k,p,q} x^{n-k} \tag{2}$$

Applying (p,q)-derivative in (2), we find

$$D_{p,q,x}G_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} G_{k,p,q} D_{p,q,x} x^{n-k}$$
$$= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} [n-k]_{p,q} p^{\binom{n-k-1}{2}} G_{k,p,q} (px)^{n-k-1} \qquad (3)$$

From (3), we have

$$D^{(1)}_{p,q,x} G_{n,p,q}(x) = [n]_{p,q} G_{n-1,p,q}(px).$$

Again using the same method as above, we have

$$D^{(2)}_{p,q,x} G_{n,p,q}(x) = \frac{[n]_{p,q}!}{[n-2]_{p,q}!} G_{n-2,p,q}(p^2 x).$$

We have the required result using mathematical induction. □

From Theorem 2.1, We note

(i) Considering $p = 1$ one holds

$$G_{n-k,q}(x) = \frac{[n-k]_q!}{[n]_q!} D^{(k)}_{q,x} G_{n,q}(x),$$

where $D^{(n)}_q$ is q-derivative, $[n]_q$ is q-number, and $G_{n,q}(x)$ is the q-Genocchi polynomials.

(ii) Considering $p = 1, q \to 1$ one holds

$$G_{n-k}(x) = \frac{n!}{(n-k)!} \frac{d^k}{dx^k} G_n(x),$$

where $G_n(x)$ is the Genocchi polynomials.

Theorem 2.2. *The (p, q)-Genocchi polynomials $G_{n,q}(x)$ satisfies the following differential equation:*

$$\frac{1}{[n]_{p,q}!} D^{(n)}_{p,q,x} G_{n,p,q}(p^{-n}x) + \frac{1}{[n-1]_{p,q}!} D^{(n-1)}_{p,q,x} G_{n,p,q}(p^{-(n-1)}x)$$
$$+ \cdots + \frac{1}{[3]_{p,q}!} D^{(3)}_{p,q,x} G_{n,p,q}(p^{-3}x) + \frac{1}{[2]_{p,q}!} D^{(2)}_{p,q,x} G_{n,p,q}(p^{-2}x)$$
$$+ D^{(1)}_{p,q,x} G_{n,p,q}(p^{-1}x) + 2G_{n,p,q}(x) - 2[n]_{p,q} p^{\binom{n-1}{2}} x^{n-1} = 0.$$

Proof. In order to find differential equation, we consider $e_{p,q}(t) \neq -1$. From the generating function of (p,q)-Genocchi polynomials, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} G_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty} p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!}+1\right)\\&=2\sum_{n=0}^{\infty} p^{\binom{n}{2}}x^n\frac{t^{n+1}}{[n]_{p,q}!}.\end{aligned}$$

By using Cauchy product, we find

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{\binom{k}{2}}G_{n-k,p,q}(x)+G_{n,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}\\&=2\sum_{n=0}^{\infty}[n]_{p,q}p^{\binom{n-1}{2}}x^{n-1}\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

From the above equation, we have

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{\binom{k}{2}}G_{n-k,p,q}(x)+G_{n,p,q}(x)\\&=2[n]_{p,q}p^{\binom{n-1}{2}}x^{n-1}.\end{aligned} \tag{4}$$

By using Theorem 2.1 in the left-side hand of (4), we find

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{\binom{k}{2}}G_{n-k,p,q}(x)+G_{n,p,q}(x)\\&=\sum_{k=0}^{n}\frac{1}{[k]_{p,q}!}D_{p,q,x}^{(k)}G_{n,p,q}(p^{-k}x)+G_{n,p,q}(x).\end{aligned} \tag{5}$$

From (4) and (5), we derive

$$\begin{aligned}&\sum_{k=0}^{n}\frac{1}{[k]_{p,q}!}D_{p,q,x}^{(k)}G_{n,p,q}(p^{-k}x)\\&+G_{n,p,q}(x)-2[n]_{p,q}p^{\binom{n-1}{2}}x^{n-1}=0.\end{aligned}$$

From the above equation, we finish the proof of Theorem 2.2. □

Corollary 2.3. *Putting $p=1$ in Theorem 2.2, one holds*

$$\begin{aligned}
&\frac{1}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(x)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(x)+\cdots\\
&+\frac{1}{[3]_q!}D_{q,x}^{(3)}G_{n,q}(x)+\frac{1}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(x)+D_{q,x}^{(1)}G_{n,q}(x)-2[n]_qx^{n-1}\\
&=0,
\end{aligned} \tag{6}$$

where $D_q^{(n)}$ is the q-derivative and $G_{n,q}(x)$ is the q-Genocchi polynomials.

Corollary 2.4. *Let $p=1, q\to 1$ in Theorem 2.2. Then, one holds*

$$\begin{aligned}
&\frac{1}{n!}\frac{d^n}{dx^n}G_n(x)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}G_n(x)+\frac{1}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}G_n(x)+\cdots\\
&+\frac{1}{3!}\frac{d^3}{dx^3}G_n(x)+\frac{1}{2!}\frac{d^2}{dx^2}G_n(x)+\frac{d}{dx}G_n(x)-2nx^{n-1}=0,
\end{aligned}$$

where $G_n(x)$ is the Genocchi polynomials.

Theorem 2.5. *The following differential equation:*

$$\begin{aligned}
&\frac{G_{n,p,q}+G_{n,p,q}(1)}{p^{\binom{n}{2}}[n]_{p,q}!}D_{p,q,x}^{(n)}G_{n,p,q}(p^{-n}x)\\
&+\frac{G_{n-1,p,q}+G_{n-1,p,q}(1)}{p^{\binom{n-1}{2}}[n-1]_{p,q}!}D_{p,q,x}^{(n-1)}G_{n,p,q}(p^{-(n-1)}x)+\cdots\\
&+\frac{G_{2,p,q}+G_{2,p,q}(1)}{p[2]_{p,q}!}D_{p,q,x}^{(2)}G_{n,p,q}(p^{-2}x)\\
&+(G_{1,p,q}+G_{1,p,q}(1))D_{p,q,x}^{(1)}G_{n,p,q}(p^{-1}x)\\
&+(G_{0,p,q}+G_{0,p,q}(1))G_{n,p,q}(x)-2[n]_{p,q}G_{n-1,p,q}(x)=0,
\end{aligned}$$

has a (p,q)-Genocchi polynomials $G_{n,p,q}(x)$ as its solution.

Proof. From $G_{n,p,q}(x)$, we have a relation as

$$\begin{aligned}
&\sum_{n=0}^{\infty}G_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\\
&=\frac{1}{2t}\left(\frac{2t}{e_{p,q}(t)+1}+\frac{2t}{e_{p,q}(t)+1}e_{p,q}(t)\right)\frac{t}{e_{p,q}(t)-1}e_{p,q}(tx)
\end{aligned}$$

$$= \frac{1}{2t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}(G_{k,p,q}+G_{k,p,q}(1))G_{n-k,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}.$$

From the equation above, we derive the following equation:

$$\begin{aligned} &2[n]_{p,q}G_{n-1,p,q}(x) \\ &=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}(G_{k,p,q}+G_{k,p,q}(1))G_{n-k,p,q}(x). \end{aligned}$$

Therefore, we complete the proof of Theorem 2.5. □

Corollary 2.6. *Setting $p = 1$ in Theorem 2.5, the following holds*

$$\begin{aligned} &\frac{G_{n,q}+G_{n,q}(1)}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(x)+\frac{G_{n-1,q}+G_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(x)+\cdots \\ &+\frac{G_{2,q}+G_{2,q}(1)}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(x)+(G_{1,q}+G_{1,q}(1))D_{q,x}^{(1)}G_{n,q}(x) \\ &+(G_{0,q}+G_{0,q}(1))G_{n,q}(x)-2[n]_qG_{n-1,q}(x)=0, \end{aligned}$$

where D_q is the q-derivative, $G_{n,q}$ is the q-Genocchi numbers, and $G_{n,q}(x)$ is the q-Genocchi polynomials.

Corollary 2.7. *Considering $p = 1, q \to 1$ in Theorem 2.5, the following holds:*

$$\begin{aligned} &\frac{G_n+G_n(1)}{n!}\frac{d^n}{dx^n}G_n(x)+\frac{G_{n-1}+G_{n-1}(1)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}G_n(x) \\ &+\cdots+\frac{G_2+G_2(1)}{2!}\frac{d^2}{dx^2}G_n(x) \\ &+(G_1+G_1(1))\frac{d}{dx}G_n(x)+(G_0+G_0(1))G_{n,q}(x)-2nG_{n-1}(x)=0, \end{aligned}$$

where G_n is the Genocchi numbers and $G_n(x)$ is the Genocchi polynomials.

Theorem 2.8. *The (p, q)-Genocchi polynomials $G_{n,p,q}(x)$ satisfies the following differential equation:*

$$\frac{\mathcal{E}_{n,p,q}+\mathcal{E}_{n,p,q}(1)}{p^{\binom{n}{2}}[n]_{p,q}!}D_{p,q,x}^{(n)}G_{n,p,q}(p^{-n}x)$$

$$+ \frac{\mathcal{E}_{n-1,p,q} + \mathcal{E}_{n-1,p,q}(1)}{p^{\binom{n-1}{2}}[n-1]_{p,q}!} D^{(n-1)}_{p,q,x} G_{n,p,q}(p^{-(n-1)}x)$$
$$+ \cdots + \frac{\mathcal{E}_{2,p,q} + \mathcal{E}_{2,p,q}(1)}{p[2]_{p,q}!} D^{(2)}_{p,q,x} G_{n,p,q}(p^{-2}x)$$
$$+ (\mathcal{E}_{1,p,q} + \mathcal{E}_{1,p,q}(1)) D^{(1)}_{p,q,x} G_{n,p,q}(p^{-1}x)$$
$$+ (\mathcal{E}_{0,p,q} + \mathcal{E}_{0,p,q}(1) - 2) G_{n,p,q}(x) = 0,$$

where $\mathcal{E}_{n,p,q}$ is the (p, q)-Euler numbers and $\mathcal{E}_{n,p,q}(x)$ is the (p, q)-Euler polynomials.

Proof. To find a differential equations with (p, q)-Euler numbers and polynomials as coefficients, we derive

$$\sum_{n=0}^{\infty} G_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{2t}{e_{p,q}(t) + 1} e_{p,q}(tx)$$
$$= \frac{1}{2} \left(\frac{2}{e_{p,q}(t) + 1} + \frac{2}{e_{p,q}(t) + 1} e_{p,q}(t) \right) \frac{t}{e_{p,q}(t) - 1} e_{p,q}(tx)$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (\mathcal{E}_{k,p,q} + \mathcal{E}_{k,p,q}(1)) G_{n-k,p,q}(x) \right) \frac{t^n}{[n]_{p,q}!}. \tag{7}$$

By using the coefficient comparison method in (7), we have

$$2G_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (\mathcal{E}_{k,p,q} + \mathcal{E}_{k,p,q}(1)) G_{n-k,p,q}(x). \tag{8}$$

Applying $D^{(k)}_{p,q,x} G_{n,p,q}(p^{-k}x) = \frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!} G_{n-k,p,q}(x)$ in (8), we find

$$\sum_{k=0}^{n} \frac{\mathcal{E}_{k,p,q} + \mathcal{E}_{k,p,q}(1)}{p^{\binom{k}{2}}[k]_{p,q}!} D^{(k)}_{p,q,x} G_{n,p,q}(p^{-k}x) - 2G_{n,p,q}(x) = 0.$$

From the above equation, we obtain the required result. □

Corollary 2.9. *Setting $p = 1$ in Theorem 2.8, one holds*

$$\frac{\mathcal{E}_{n,q} + \mathcal{E}_{n,q}(1)}{[n]_q!} D^{(n)}_{q,x} G_{n,q}(x)$$

$$+\frac{\mathcal{E}_{n-1,q}+\mathcal{E}_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(x)$$
$$+\cdots+\frac{\mathcal{E}_{2,q}+\mathcal{E}_{2,q}(1)}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(x)+(\mathcal{E}_{1,q}+\mathcal{E}_{1,q}(1))D_{q,x}^{(1)}G_{n,q}(x)$$
$$+(\mathcal{E}_{0,q}+\mathcal{E}_{0,q}(1)-2)G_{n,q}(x)=0,$$

where $D_q^{(n)}$ *is the q-derivative,* $\mathcal{E}_{n,q}$ *is the q-Euler numbers, and* $\mathcal{E}_{n,q}(x)$ *is the q-Euler polynomials.*

Corollary 2.10. *Setting* $p=1, q\to 1$ *in Theorem 2.8, one holds*

$$\frac{\mathcal{E}_n+\mathcal{E}_n(1)}{n!}\frac{d^n}{dx^n}G_n(x)+\frac{\mathcal{E}_{n-1}+\mathcal{E}_{n-1}(1)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}G_n(x)+\cdots$$
$$+\frac{\mathcal{E}_2+\mathcal{E}_2(1)}{2!}\frac{d^2}{dx^2}G_n(x)+(\mathcal{E}_1+\mathcal{E}_1(1))\frac{d}{dx}G_n(x)$$
$$+(\mathcal{E}_0+\mathcal{E}_0(1)-2)G_n(x)=0,$$

where $\mathcal{E}_n$ *is the Euler numbers and* $\mathcal{E}_n(x)$ *is the Euler polynomials.*

3. Conclusion

We found some differential equation by using a relationship between (p,q)-Genocchi numbers and polynomials. We also obtained relationship between Genocchi, q-Genocchi, and (p,q)-Genocchi polynomials. Since Genocchi polynomials are useful in various fields, it is hoped that constructing degenerate q-Genocchi polynomials that cannot be found at present and finding their properties could be useful research.

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